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# SIMULATION OF THE PHASE OF ESTABLISHMENT OF THE RHEOMETRIC FLOW OF A VISCOELASTIC FLUID

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*The flow between two plates in the phase of establishment for the case of a viscoelastic fluid in comparison with a Newtonian fluid is investigated numerically and analytically. The quantitative and qualitative features of the flow are analyzed. It is noted that in a number of cases this flow has wave features, and it is determined that the presence of viscoelasticity usually increases the time of establishment of the flow. It is pointed out that some of the revealed features of the flow can be used in experimental investigations of non-Newtonian fluids.*

As is well known, the correct description of the transient regimes of flow of viscoelastic fluids is an important problem of rheology [1, p. 73]. To simulate the process of establishment of a fluid flow between two plates, in the present work we used the Olroyd contravariant model [2], which in many cases describes well a wide class of liquids, including high-elastic ones [3].

**Formulation of the Problem.** We will assume the state of rest as the *initial condition*. The *boundary conditions* are as follows: the lower plate is at rest; the process of smooth acceleration of the upper plate from zero velocity to the asymptotic velocity $U$ will be described by the relation $\frac{\alpha^* t^*}{1 + \alpha^* t^*} U$. It is obvious that in the nondimensional variables

$$u(t, 1) = \frac{\alpha t}{1 + \alpha t}$$

the parameter $\alpha$ characterizes the acceleration rate of the plate.

For an Olroyd fluid with the equation of state [1, 2]

$$\theta \frac{D_0 \boldsymbol{\sigma}}{D_0 t} + \boldsymbol{\sigma} = \nu \mathbf{d} ,$$

where

$$\frac{D_0 \sigma^{ij}}{D_0 t} = \frac{\partial \sigma^{ij}}{\partial t} + v^k \frac{\partial \sigma^{ij}}{\partial x^k} - \frac{\partial \nu^i}{\partial x^k} \sigma^{kj} - \frac{\partial \nu^j}{\partial x^k} \sigma^{ik} ,$$

and assuming that the transverse velocity component is equal to 0 and all the flow characteristics remain constant along the plates, we have

$$\theta \frac{\partial \sigma^{yy}}{\partial t} + \sigma^{yy} = 0 ,$$

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TABLE 1. Parameters Used in Investigation of the Flow for Specific Fluids

| Fluid | μ, Pa·sec | Re |
|---|---|---|
| Water | $10^{-3}$ | 100 |
| Glycerin | 1.5 | 0.067 |
| High-pressure polyethylene | $\cong 10^{5}$ | $\cong 10^{-6}$ |

$$\sigma^{yy}(t, y) = \sigma^{yy}(0, y) \exp(-t/\theta) .$$

Since at the initial instant the fluid is at rest, $\sigma^{yy}(0, y) = 0$ and $\sigma^{yy}(t, y) = 0$, and, obviously, the equations of motion and rheological state are reduced to a system that corresponds to the flow with longitudinal velocity $u = u(t, y)$:

$$\frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial y} \tag{1}$$

$$\frac{\partial \sigma}{\partial t} + \frac{\sigma}{\mathrm{W}} = \frac{1}{\mathrm{Re\,W}} \frac{\partial u}{\partial y} \tag{2}$$

with the conditions

$$u(0, y) = u(t, 0) = 0 \tag{3}$$

$$u(t, 1) = \frac{\alpha t}{1 + \alpha t} , \tag{4}$$

where Re and W are the Reynolds and Weissenberg numbers.

**Solution and Discussion of the Results.** Problem (1)–(4) was solved by the finite-element method using quadratic Lagrangian finite elements [4]. The time derivatives were approximated following the Crank–Nicholson scheme. The discrete systems of equations were solved by the method of successive over-relaxation [5].

For an actual rheometric experiment, the quantities $H = 0.5$ cm and $U = 2$ cm/sec should be considered to be appropriate; next, we will agree to consider the time in which the acceleration of the upper plate decreases from the initial acceleration (it was selected from considerations of an optimum count time) to a fraction of $\varepsilon = 1.5 \cdot 10^{-3}$ as the *time of establishment* of the motion of the upper plate. From Eq. (4) we obtain that this time is $T_{\mathrm{est}}^{\mathrm{pl}} \cong 24.8$; if $\alpha^* = 4$ sec$^{-1}$, then for the dimensional time $T_{\mathrm{est}}^{\mathrm{pl}^*} \cong 6$ sec, which is acceptable from the practical point of view. The parameters needed for three fluids under these conditions are presented in Table 1. Clearly, in order to describe flows of a wide range of fluids, i.e., from virtually Newtonian fluids (water) to substantially viscoelastic ones (melts of polymers), it is necessary to prescribe the range of Re from $10^{-6}$ to 199. The change in the Weissenberg number W for the selected values of $U$ and $H$ was roughly chosen from 0.04 to 400 in accordance with [6, p. 5].

For the viscoelastic fluid the distributions of the velocities and stresses frequently had a rather complex shape very different from the Newtonian-fluid profiles, which in the flow under consideration are regular and convex to the left. For example, concerning the left profile (shown in Fig. 1) we can say that the flow is of a substantially evolutionary nature: "nothing is known yet" in the lower portion of the gap about the already rather developed motion in the upper portion of the gap. This allows an assumption that similar solutions are partly of a wave nature.

To confirm this assumption, we note that system (1)–(4) yields the equation

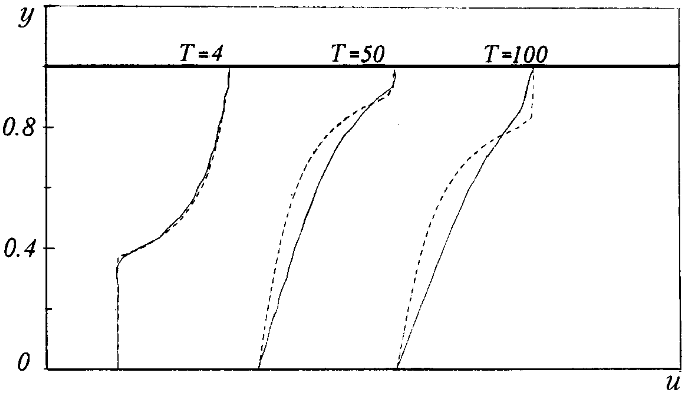

Fig. 1. Flow profiles of a viscoelastic fluid for Re = 1 and W = 40 for three instants of time in comparison with the corresponding solutions of the wave equation (the dashed curves).

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\mathrm{W}} \frac{\partial u}{\partial t} = \frac{1}{\mathrm{W\,Re}} \frac{\partial^2 u}{\partial y^2}, \tag{5}$$

which for large W or small products of W and Re changes to a wave equation with the phase velocity:

$$c = \frac{1}{\sqrt{W \,\mathrm{Re}}}.$$

It is not difficult to construct the solution of the wave equation that satisfies conditions (3) and (4):

$$u_{\mathrm{w}}(t, y) = V\left(t - \frac{1-y}{c}\right) - V\left(t - \frac{1+y}{c}\right), \tag{6}$$

where

$$V(p) = \begin{cases} 0 & \text{for } p \le 0, \\ V_n(p_n) & \text{for } p > 0. \end{cases} \tag{7}$$

Here $n$ is the integral part of the ratio $p/T$; $T = 2/c$ is the time of double passage of the gap by a wave. The function $V_n(p_n)$ is determined using the recurrence relations

$$p_0 = p - nT, \quad V_0 = \frac{\alpha p_0}{1 + \alpha p_0}; \quad p_k = p_{k-1} + T, \quad V_k = V_{k-1} + \frac{\alpha p_k}{1 + \alpha p_k}, \quad k = 1, \ldots, n. \tag{8}$$

The solutions of the wave equation are given in Fig. 1 (the dashed curves) in comparison with those of system (1)–(4). One can observe that the wave profile has discontinuities. Analyzing the solution (6)–(8), one may conclude that the points of discontinuity of the first derivative must exist on at least certain time intervals. Strictly speaking, the solution is redefined to these points by continuity.

The similarity of two solutions is especially evident for the very early instant. Based on a sufficiently large amount of calculations performed, it is possible to draw the conclusion that even for not-too-large W and not-too-small W and Re (of one order of magnitude) the solution of system (1)–(4) is frequently similar to the wave solution (6)–(8). At later stages the flow tends to take the established form with a direct profile (the second and third profiles in Fig. 1); however, it continues to retain certain features of the wave solution, in particular, the points of discontinuity of its derivative correspond to the segments of the large curvature of

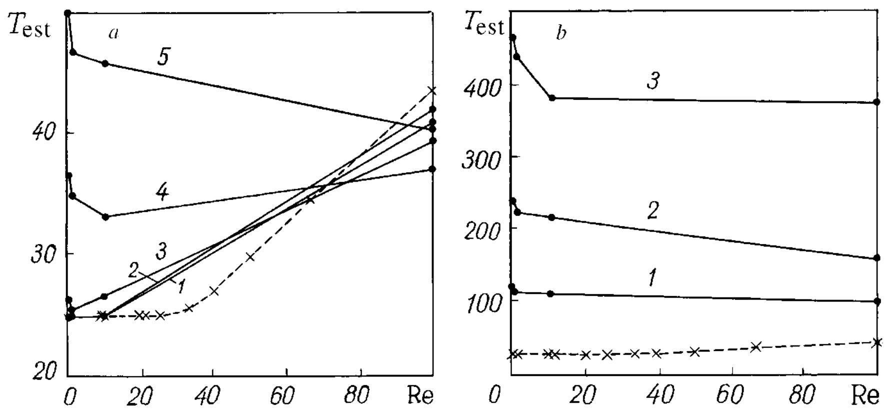


Fig. 2. Dependences of the time of establishment of the flow of a viscoelastic fluid on the Reynolds number in comparison with the dependence for a Newtonian fluid (the dashed curves) for the values of the Weissenberg number (in the order of numbering of the curves): a) 0.1; 1; 2; 3; 4; b) 10; 20; 40.

the flow profile that move up and down between the plates. It can be assumed that in the case of flow of a polymeric fluid the presence of such segments of anomalously large shear deformation can produce mechanodestruction at the stage of establishment of the flow.

Since the distribution of the stress $\sigma$ has a similar form (for it one can obtain an equation analogous to Eq. (5)), proceeding from the aforesaid, one could, probably, observe in the experiment periodic more or less sharp changes in the force acting on the upper plate during its motion in the phase of establishment and evaluate the magnitude of W and Re by the period of these "changes" and in the end the relaxation time.

We assume that one of the most important characteristics of the flow is the time needed for reaching the stationary regime compared to the time $T_{est}^{pl}$. This time was determined from the condition that at this instant the norm of the time derivative of the solution $\max\limits_{k} \left|\frac{\partial w_k^n}{\partial t}\right|$ does not exceed $\varepsilon$ ($w_k^n$ is the value of the velocity or stress at the $k$th node of finite-element splitting on the $n$th time layer).

The results of determination of the above quantity for $\alpha = 1$ and $\mathrm{Re} \geq 1$ in comparison with the case of the Newtonian fluid [5] are shown in Fig. 2. It can be seen that in the presence of viscoelasticity the time of establishment in the main increases; here this increase is more substantial for not very large Reynolds numbers. As the Weissenberg number grows, the time of establishment increases. As to the dependences of the time on the Reynolds number for constant W, generally speaking, they are not monotonically increasing dependences as could be expected. Apparently, this is explained by the rather complex nature of the flow in the phase of establishment, which has been discussed above.

For very small Reynolds numbers (to $10^{-6}$), which is timely for viscous high-molecular-weight fluids, the convergence of the process was retained only on assignment of a very small time step, so that it was virtually impossible to calculate the process of establishment to completion. However, it is obvious from Eq. (5) that the solution in this case must give a profile close to the linear one and an establishment time close to $T_{est}^{pl}$.

Similar difficulties did not arise for $0.1 \leq \mathrm{Re} \leq 100$ and $\mathrm{W} \geq 200$. But here, too, we failed to reach the criterion of establishment. Since the solution is close to the wave solution (6)–(8), while the latter, as can easily be verified, is not to be established, such fluids can have at least a large time of establishment. The question of whether the process of flow of these fluids can reach the steady state at all remains open.

Finally, for the dependence of the establishment time on the parameter of acceleration intensity $\alpha$ (which varied in the range 0.5–5), we note that for small W and Re it varies in proportion to $\alpha$; at the same time, in the region of W > 10 and Re > 20 the time is almost independent of $\alpha$, so that the process of flow is mainly determined by the properties of the fluid itself, rather than by the nature of the motion of the plate (in the investigated range of $\alpha$).

Summing up the present investigation, we can conclude that the phase of establishment of the flow of a viscoelastic fluid in comparison with the flow of a Newtonian fluid is characterized by some qualitative features, which can be used in experimental studies of the properties of specific liquids.

## NOTATION

$U$, asymptotic velocity of motion of the upper plate; $H$, width of the gap between the plates; $t^*$, dimensional time; $u$, $y$, and $t$, dimensionless values of the longitudinal flow velocity, transverse coordinate, and time (on a scale of $U$, $H$, and $H/U$, respectively); $\alpha^*$ and $\alpha = \alpha^* H/U$, dimensional and dimensionless parameters that characterize the acceleration rate of the upper plate; $\boldsymbol{\sigma}$ and $\mathbf{d}$, tensors of stresses and deformation rates (with the superscripts they are the contravariant components of these tensors); $x$ and $v$ with the superscripts are the components of the coordinates and velocities; $\theta$, relaxation time in the Olroyd rheological model; $\sigma$, tangential stress; $\mu$ and $\nu$, dynamic and kinematic viscosities; $\mathrm{Re} = UH/\nu$, Reynolds number; $\mathrm{W} = \theta U/H$, Weissenberg number; $T_{\mathrm{est}}^{\mathrm{pl}*}$ and $T_{\mathrm{est}}^{\mathrm{pl}}$, dimensional and dimensionless times of establishment of the motion of the upper plate; $\varepsilon$, small parameter that determines the times when the motion of the upper plate and the fluid reach the stationary regime; c, velocity of propagation of the shear wave; $u_{\mathrm{w}}$, solution of the wave equation given by expressions (6)–(8); $V$, $p$, and $T$, parameters used for representation of the wave solution (6)–(8).

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